# Superintegrability of rational Ruijsenaars-Schneider systems and their action-angle duals

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**Abstract**

We explain that the action-angle duality between the rational Ruijsenaars-Schneider and hyperbolic Sutherland systems implies immediately the maximal superintegrability of these many-body systems. We also present a new direct proof of the Darboux form of the reduced symplectic structure that arises in the 'Ruijsenaars gauge' of the symplectic reduction underlying this case of action-angle duality. The same arguments apply to the $BC_n$ generalization of the pertinent dual pair, which was recently studied by Pusztai developing a method utilized in our direct calculation of the reduced symplectic structure.

# 1 Introduction

The subject of superintegrability can be regarded as an offspring of the Kepler problem, which is 'more integrable' than motion in an arbitrary spherically symmetric potential due to the existence of the extra conserved quantities provided by the Runge-Lenz vector. Recently we witnessed intense studies of superintegrable dynamical systems motivated partly by interesting examples and partly by the natural goal to classify systems with nice properties. See, for example, [2, 6, 15] and references therein.

Let us briefly recall the relevant notions of integrability for a Hamiltonian system $(M, \omega, H)$ living on a $2n$-dimensional symplectic manifold. Such a system is called *Liouville integrable* if there exist $n$ independent functions $h_i \in C^\infty(M)$ $(i = 1, \ldots, n)$ that are in involution with respect to the Poisson bracket and the Hamiltonian can be written as $H = \mathcal{H}(h_1, ..., h_n)$ through some smooth function $\mathcal{H}$ of $n$ variables. Importantly, one has to require also that the flows of the $h_i$ are all complete. A Liouville integrable system $(M, \omega, H)$ is termed *maximally superintegrable* if it admits $(n-1)$ additional constants of motion, say $f_j \in C^\infty(M)$, such that $h_1, \ldots, h_n, f_1, \ldots, f_{n-1}$ are functionally independent[1]. The generic trajectories of $(M, \omega, H)$ are then given by the connected components of the 1-dimensional joint level surfaces of the $(2n-1)$ constants of motion. As a consequence, those trajectories of $(M, \omega, H)$ that stay inside some compact submanifold of $M$ are necessarily homeomorphic to the circle, since they are connected and compact 1-dimensional manifolds. This implies that Liouville integrable systems having compact Liouville tori are rarely superintegrable, because their trajectories are usually not closed. On the other hand, it is common knowledge, supported by rigorous results [5], that systems describing repulsive interactions of particles are superintegrable. Concretely, the scattering data provided by the asymptotic particle momenta and differences of their conjugates yield sufficiently many constants of motion. More abstractly [16], the classical wave maps furnish symplectomorphisms to obviously superintegrable free systems.

The aim of this contribution is to explain the superintegrability of the celebrated rational Ruijsenaars-Schneider [13] and hyperbolic Sutherland systems [14, 3] in a self-contained manner. Since these one-dimensional many-body systems support factorizable scattering [12], their superintegrability is not surprising. However, we shall not use any scattering theory argument, which usually requires non-trivial analysis of the dynamics. Instead of scattering theory, we shall directly rely on special features of the 'action-angle maps' of these Liouville integrable systems. Indeed, it is known that these two systems form a dual pair in the sense that they live on symplectomorphic phase spaces, and the particle-positions of each one of the two systems serve as action-variables of the other system. The duality property was discovered by Ruijsenaars [12] in his direct construction of 'action-angle maps' that realize the introduction of action-angle variables. More recently [4], this duality has been fitted into the geometric framework of symplectic reduction, which we shall utilize for showing superintegrability.

In Section 2, based on [1], we recall the elementary observation that Liouville integrable systems admitting global action-angle maps of maximally non-compact type are maximally superintegrable. Then, in Section 3, we explain how the geometric picture behind the rational Ruijsenaars-Schneider and hyperbolic Sutherland systems permits to see easily that their action-angle maps are the inverses of each other and are of maximally non-compact type. In Section

[1]Below the term superintegrable will always mean maximally superintegrable.

4, we point out that this mechanism applies also to the generalized Ruijsenaars-Schneider and Sutherland systems that are associated with the $BC_n$ root system. The $BC_n$ generalization of the pertinent dual action-angle maps was recently developed by Pusztai [10, 11]. In Appendix A, we take the opportunity to apply his ideas for improving the previous (correct but not self-contained) calculation of the reduced symplectic structure given in [4].

# 2 Action-angle maps of maximally non-compact type

In scattering systems the canonical conjugates of the actions run over the line. Later we shall exhibit interesting examples where the canonical transformation to these Darboux variables represents an action-angle map of maximally non-compact type as defined below.

Consider a Liouville integrable Hamiltonian system $(M, \omega, H)$ possessing the $n$ Poisson commuting, independent constants of motion $h_i \in C^\infty(M)$, $i = 1, \ldots, n$. Let us assume that globally well-defined action-variables with globally well-defined canonical conjugates exist. By definition, this means that there exists a phase space $(\hat{M}, \hat{\omega})$ of the form

$$\hat{M} = \mathcal{C}_n \times \mathbb{R}^n = \{(\hat{p}, \hat{q}) \,|\, \hat{p} \in \mathcal{C}_n,\ \hat{q} \in \mathbb{R}^n\} \tag{1}$$

with a connected open domain $\mathcal{C}_n \subseteq \mathbb{R}^n$ and canonical symplectic form

$$\hat{\omega} = \sum_{i=1}^{n} d\hat{q}_i \wedge d\hat{p}_i, \tag{2}$$

which is symplectomorphic to $(M, \omega)$ and permits identification of the Hamiltonians $h_i$ as functions of the action-variables $\hat{p}_j$. More precisely, we assume the existence of a symplectomorphism

$$A : M \to \hat{M} \tag{3}$$

such that the functions $h_i \circ A^{-1}$ do not depend on $\hat{q}$ and

$$X_{i,j} := \frac{\partial h_i \circ A^{-1}}{\partial \hat{p}_j} \tag{4}$$

yields an invertible matrix $X(\hat{p})$ at every $\hat{p} \in \mathcal{C}_n$. As in [1], the map $A$ is referred to as a *global action-angle map of maximally non-compact type.* The target $(\hat{M}, \hat{\omega})$ of $A$ is often called the action-angle phase space of the system $(M, \omega, H)$.

To clarify our conventions, note that for any real function $F \in C^\infty(\hat{M})$ the Hamiltonian vector field $\boldsymbol{X}_F$ is here defined by

$$dF = \hat{\omega}(\,\cdot\,, \boldsymbol{X}_F), \tag{5}$$

and the Poisson bracket of two functions $F_1, F_2$ reads

$$\{F_1, F_2\}_{\hat{M}} = dF_1(\boldsymbol{X}_{F_2}) = \hat{\omega}(\boldsymbol{X}_{F_2}, \boldsymbol{X}_{F_1}). \tag{6}$$

In particular, we have

$$\{\hat{p}_j, \hat{q}_k\}_{\hat{M}} = \delta_{j,k}, \quad \{\hat{p}_j, \hat{p}_k\}_{\hat{M}} = \{\hat{q}_j, \hat{q}_k\}_{\hat{M}} = 0. \tag{7}$$

If a global action-angle map of maximally non-compact type exists, then one can introduce functions $f_i \in C^\infty(M)$ $(i = 1, \dots, n)$ by the definition

$$(f_i \circ A^{-1})(\hat{p}, \hat{q}) := \sum_{j=1}^{n} \hat{q}_j X(\hat{p})^{-1}_{j,i} \quad \text{with} \quad \sum_{j=1}^{n} X(\hat{p})_{i,j} X(\hat{p})^{-1}_{j,k} = \delta_{i,k}. \tag{8}$$

By using that $A$ is a symplectomorphism, one obtains the Poisson brackets

$$\{h_i, f_j\}_M = \delta_{i,j}, \quad \{f_i, f_j\}_M = 0. \tag{9}$$

Indeed, the first relation is immediate from $\{h_i \circ A^{-1}, f_j \circ A^{-1}\}_{\hat{M}} = \sum_{k=1}^{n} \frac{\partial h_i \circ A^{-1}}{\partial \hat{p}_k} \frac{\partial f_j \circ A^{-1}}{\partial \hat{q}_k}$, and the second relation is also easily checked. Together with $\{h_i, h_j\}_M = 0$, (9) implies that the $2n$ functions $h_1, \dots, h_n, f_1, \dots, f_n$ are functionally independent at every point of $M$.

It is plain that the choice of any of the $2n$ functions $h_1, \dots, h_n, f_1, \dots, f_n$ as the Hamiltonian yields a maximally superintegrable system. For example, the $(2n-1)$ independent functions $h_1, \dots, h_n, f_1, \dots, f_{n-1}$ Poisson commute with $h_n$. Under mild conditions, it can be shown [1] that the generic Liouville integrable Hamiltonian of the form $H = \mathcal{H}(h_1, \dots, h_n)$ is also maximally superintegrable.

# 3 Hyperbolic Sutherland and rational RS systems

We below explain that the hyperbolic Sutherland system and the rational Ruijsenaars-Schneider system admit global action-angle maps of maximally non-compact type, which implies their maximal superintegrability through the simple construction presented in the previous section. Remarkably, the pertinent two action angle-maps are the inverses of each other.

## 3.1 Definition of the systems

The hyperbolic Sutherland system [14, 3] lives on the phase space

$$M := \mathcal{C}_n \times \mathbb{R}^n = \{(q,p) \,|\, q \in \mathcal{C}_n,\, p \in \mathbb{R}^n\} \tag{10}$$

with the domain

$$\mathcal{C}_n = \{q \in \mathbb{R}^n \,|\, q_1 > q_2 > \dots > q_n\}. \tag{11}$$

The symplectic form is the canonical one

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j. \tag{12}$$

A family of $n$ independent commuting Hamiltonians is given by

$$h_k(q,p) := \mathrm{tr}(L(q,p)^k), \qquad k = 1, \dots, n, \tag{13}$$

where $L(q,p)$ is the $n \times n$ Hermitian Lax matrix having the entries

$$L(q,p)_{j,k} := p_j \delta_{j,k} + \mathrm{i}(1 - \delta_{j,k}) \frac{\kappa}{\sinh(q_j - q_k)}, \tag{14}$$

using a non-zero real parameter $\kappa$. The flows of the $h_k$ are complete, and the main Hamiltonian of interest is

$$H(q,p) := \frac{1}{2}h_2(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{1\le j<k\le n} \frac{\kappa^2}{\sinh^2(q_j - q_k)}. \tag{15}$$

Thus $q_i$ ($i = 1,\dots,n$) can be interpreted as the positions of $n$ interacting particles moving on the line, restricted to the domain $\mathcal{C}_n$ by energy conservation.

The rational Ruijsenaars-Schneider (RS) system [13] lives on the same phase space, but for later purpose we now denote the phase space points as pairs $(\hat{p},\hat{q})$. That is, the RS phase space is the symplectic manifold $(\hat{M},\hat{\omega})$ with[2]

$$\hat{M} := \mathcal{C}_n \times \mathbb{R}^n = \{(\hat{p},\hat{q}) \,|\, \hat{p}\in\mathcal{C}_n,\ \hat{q}\in\mathbb{R}^n\}, \qquad \hat{\omega} = \sum_{j=1}^{n} d\hat{q}_j \wedge d\hat{p}_j. \tag{16}$$

Now a basic set of Liouville integrable Hamiltonians is provided by $\hat{h}_l \in C^\infty(\hat{M})$ for $l = 1,\dots,n$, where we define

$$\hat{h}_l(\hat{p},\hat{q}) := \mathrm{tr}(\hat{L}(\hat{p},\hat{q})^l), \qquad \forall l\in\mathbb{Z}. \tag{17}$$

Here, $\hat{L}$ is the (positive definite) RS Lax matrix having the entries

$$\hat{L}(\hat{p},\hat{q})_{j,k} := u_j(\hat{p},\hat{q}) \left[\frac{2\mathrm{i}\kappa}{2\mathrm{i}\kappa + (\hat{p}_j - \hat{p}_k)}\right] u_k(\hat{p},\hat{q}), \tag{18}$$

where the $\mathbb{R}_+$-valued functions $u_j(\hat{p},\hat{q})$ are given by

$$u_j(\hat{p},\hat{q}) := e^{-\hat{q}_j} z_j(\hat{p})^{\frac{1}{2}} \quad \text{with} \quad z_j(\hat{p}) := \prod_{\substack{m=1\\(m\neq j)}}^{n} \left[1 + \frac{4\kappa^2}{(\hat{p}_j - \hat{p}_m)^2}\right]^{\frac{1}{2}}. \tag{19}$$

In our convention, the principal RS Hamiltonian $\hat{H} = \frac{1}{2}(\hat{h}_1 + \hat{h}_{-1})$ reads

$$\hat{H}(\hat{p},\hat{q}) = \sum_{k=1}^{n} (\cosh 2\hat{q}_k) \prod_{\substack{j=1\\(j\neq k)}}^{n} \left[1 + \frac{4\kappa^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}, \tag{20}$$

and can be viewed as describing $n$ interacting 'particles' with *positions* $\hat{p}_k$ ($k = 1,\dots,n$).

## 3.2 Dual gauge slices in symplectic reduction

Ruijsenaars [12] discovered an intriguing duality relation between the pertinent two integrable many-body systems, which he called action-angle duality. Next we recall the geometric interpretation of this duality, nowadays also called 'Ruijsenaars duality', following the joint work of Klimčík with one of us [4].

Let $G$ denote the real Lie group $GL(n,\mathbb{C})$ and identify the dual space of the corresponding real Lie algebra $\mathfrak{g} := gl(n,\mathbb{C})$ with itself using the invariant bilinear form

$$\langle X, Y\rangle := \Re\mathrm{tr}(XY), \qquad \forall X,Y\in\mathfrak{g}. \tag{21}$$

[2]The notation anticipates that $(\hat{M},\hat{\omega})$ is the action-angle phase space of the Sutherland system $(M,\omega,H)$.

Consider the minimal coadjoint orbit $\mathcal{O}_\kappa$ of the group $U(n)$ given as a set by

$$\mathcal{O}_\kappa := \{\mathrm{i}\kappa(vv^\dagger - \mathbf{1}_n) \,|\, v \in \mathbb{C}^n,\ |v|^2 = n\}. \tag{22}$$

Here $v$ is viewed as a column vector, we identified $\mathfrak{u}(n)$ with its dual by the restriction of the scalar product (21), and shall also use the notation

$$\zeta(v) := \mathrm{i}\kappa(vv^\dagger - \mathbf{1}_n). \tag{23}$$

Trivializing $T^*G$ by means of left-translations, we introduce the 'extended cotangent bundle'

$$P^{\mathrm{ext}} := T^*G \times \mathcal{O}_\kappa \equiv G \times \mathfrak{g} \times \mathcal{O}_\kappa = \{(g, J, \zeta) \,|\, g \in G,\ \ J \in \mathfrak{g},\ \ \zeta \in \mathcal{O}_\kappa\}. \tag{24}$$

The symplectic form of $P^{\mathrm{ext}}$ can be written as

$$\Omega^{\mathrm{ext}} = d\langle J, g^{-1}dg\rangle + \Omega^{\mathcal{O}_\kappa} \tag{25}$$

where $\Omega^{\mathcal{O}_\kappa}$ is the standard (Kirillov-Kostant-Souriau) symplectic form of $\mathcal{O}_\kappa$.

Our basic tool is symplectic reduction of $(P^{\mathrm{ext}}, \Omega^{\mathrm{ext}})$ by the group

$$K := U(n) \times U(n) \tag{26}$$

acting via the symplectomorphisms

$$\Psi_{\eta_L,\eta_R}(g, J, \zeta) := (\eta_L g \eta_R^{-1}, \eta_R J \eta_R^{-1}, \eta_L \zeta \eta_L^{-1}), \qquad \forall(\eta_L, \eta_R) \in K. \tag{27}$$

The momentum map $\Phi : P^{\mathrm{ext}} \to \mathfrak{u}(n) \oplus \mathfrak{u}(n)$ that generates this action is given by

$$\Phi(g, J, \zeta) = ((gJg^{-1})_{\mathfrak{u}(n)} + \zeta, -J_{\mathfrak{u}(n)}), \tag{28}$$

where $X_{\mathfrak{u}(n)} = \frac{1}{2}(X - X^\dagger)$ is the anti-Hermitian part of any $X \in \mathfrak{g}$. The reduction is defined by setting the momentum map to zero. The associated reduced phase space

$$P^{\mathrm{red}} := \Phi^{-1}(0)/K \tag{29}$$

turns out to be a smooth symplectic manifold, with reduced symplectic form $\Omega^{\mathrm{red}}$. The point is that the $K$-orbits in the 'constraint surface' $\Phi^{-1}(0)$ admit two global cross sections that give rise to natural identifications of the reduced phase space $(P^{\mathrm{red}}, \Omega^{\mathrm{red}})$ with the Sutherland phase space $(M, \omega)$ and the RS phase space $(\hat{M}, \hat{\omega})$, respectively.

The first cross section is the so-called 'Sutherland gauge slice' $S \subset \Phi^{-1}(0)$ defined by

$$S := \{\, (e^{\boldsymbol{q}}, L(q,p), \zeta(v_0)) \,|\, (q,p) \in M \,\}, \tag{30}$$

where $\boldsymbol{q} := \mathrm{diag}(q_1, \ldots, q_n)$ and every component of $v_0 \in \mathbb{C}^n$ is equal to 1. In fact, $S$ intersects every $K$-orbit in $\Phi^{-1}(0)$ precisely once, and with the tautological embedding $\iota_S : S \to P^{\mathrm{ext}}$ it satisfies

$$\iota_S^*(\Omega^{\mathrm{ext}}) = \sum_{k=1}^n dp_k \wedge dq_k = \omega. \tag{31}$$

By its very definition (30), $S$ can be identified with $M$, and the last equation permits to view $(M,\omega)$ as a model of the reduced phase space $(P^{\rm red},\Omega^{\rm red})$.

An alternative model of $(P^{\rm red},\Omega^{\rm red})$ is furnished by the following 'Ruijsenaars gauge slice'

$$\hat{S} := \{\, (\hat{L}(\hat{p},\hat{q})^{\frac{1}{2}}, \hat{\boldsymbol{p}}, \zeta(v(\hat{p},\hat{q})))\,|\, (\hat{p},\hat{q})\in \hat{M}\,\}, \tag{32}$$

where $\hat{\boldsymbol{p}} = \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$ and $v(\hat{p},\hat{q})$ is the vector of squared-norm $n$ given by

$$v(\hat{p},\hat{q}) := \hat{L}(\hat{p},\hat{q})^{-\frac{1}{2}} u(\hat{p},\hat{q}), \tag{33}$$

using the Lax matrix $\hat{L}$ and the vector $u$ introduced in eqs. (18-19). In fact, $\hat{S}$ also intersects every $K$-orbit in $\Phi^{-1}(0)$ precisely once, and with the tautological embedding $\iota_{\hat{S}}: \hat{S}\to P^{\rm ext}$ it verifies

$$\iota_{\hat{S}}^*(\Omega^{\rm ext}) = \sum_{k=1}^n d\hat{q}_k\wedge d\hat{p}_k = \hat{\omega}. \tag{34}$$

Thus, identifying $\hat{S}$ (32) with $\hat{M}$, we see that $(\hat{M},\hat{\omega})$ also represents a model of the reduced phase space $(P^{\rm red},\Omega^{\rm red})$. It will be clear shortly that the two gauge slices $S$ and $\hat{S}$ are dual to each other in the sense that they geometrically engender Ruijsenaars' action-angle duality between the Sutherland and the RS systems.

The equality (31) goes back to [7] and equality (34) was first proved in [4]. The proof presented in [4] uses the 'external information' that the eigenvalues of $\hat{L}$ form an Abelian Poisson algebra under the Darboux structure $\hat{\omega}$. A completely self-contained direct proof of (34) will be given in the appendix of the present communication.

## 3.3 Action-angle duality and superintegrability

In the previous subsection we described the equivalences

$$(M,\omega) \longleftrightarrow (S,\iota_S^*(\Omega^{\rm ext})) \longleftrightarrow (P^{\rm red},\Omega^{\rm red}) \longleftrightarrow (\hat{S},\iota_{\hat{S}}^*(\Omega^{\rm ext})) \longleftrightarrow (\hat{M},\hat{\omega}). \tag{35}$$

By composing the relevant maps, we obtain a symplectomorphism $A: M\to\hat{M}$, $A^*(\hat{\omega})=\omega$. It follows easily from the geometric picture that the map $A$ operates according to the rule

$$A: (q,p)\mapsto (\hat{p},\hat{q}) \tag{36}$$

characterized the property

$$(\hat{L}(\hat{p},\hat{q})^{\frac{1}{2}}, \hat{\boldsymbol{p}}, \zeta(v(\hat{p},\hat{q}))) = (\eta(q,p)e^{\boldsymbol{q}}\eta(q,p)^{-1}, \eta(q,p)L(q,p)\eta(q,p)^{-1}, \eta(q,p)\zeta(v_0)\eta(q,p)^{-1}), \tag{37}$$

where $\eta(q,p)\in U(n)$ is uniquely determined up to right-multiplication by a scalar matrix.

Now we are ready to harvest consequences of the above construction. When doing so, we view $q_i,p_i$ and $\hat{p}_i,\hat{q}_i$ as evaluation functions on $M$ and on $\hat{M}$, respectively. The following statements are readily checked:

- First, the particle-positions $\hat{p}_i$ of the RS system are converted by the map $A$ into action-variables $\hat{p}_i \circ A$ of the Sutherland system, and at the same time the canonical momenta $\hat{q}_i$ of the RS system are converted into the corresponding non-compact 'angle-variables' $\hat{q}_i \circ A$. This statement holds since $(\hat{p}_i \circ A)(q,p)$ are just the ordered eigenvalues of the Sutherland Lax matrix $L(q,p)$. In short, the RS particle-positions and their conjugates play the roles of Sutherland action-variables and their conjugates.
- Second, since the functions $e^{2q_i} \circ A^{-1}$ on $\hat{M}$ are just the ordered eigenvalues of the RS Lax matrix $\hat{L}$, we see that the Sutherland particle-positions $q_i$ are converted by $A^{-1}$ into action-variables $q_i \circ A^{-1}$ of the RS system, and the Sutherland momenta $p_i$ are converted into the non-compact angle-variables $p_i \circ A^{-1}$ of the RS system. That is, the Sutherland particle-positions and their conjugates play the roles of RS action-variables and their conjugates.
- Third, the maps $A$ and $A^{-1}$ are global action-angle maps of maximally non-compact type in the sense defined in Section 2.

To verify the third property for the map $A$, one has to consider the commuting Hamiltonians $h_k$ of equation (13), which on the action-angle phase space $\hat{M}$ take the form

$$(h_k \circ A^{-1})(\hat{p}, \hat{q}) = \sum_{l=1}^{n} \hat{p}_l^k. \tag{38}$$

It is easily found from the Vandermonde-determinant formula that

$$\det\left(\frac{\partial h_k \circ A^{-1}}{\partial \hat{p}_j}\right) = n! \prod_{1 \leq i < j \leq n} (\hat{p}_j - \hat{p}_i). \tag{39}$$

This never vanishes on the domain $\mathcal{C}_n$, proving the claim. As for $A^{-1}$, notice from (17) and (37) that

$$(\hat{h}_k \circ A)(q,p) = \sum_{l=1}^{n} e^{2kq_l}, \qquad \forall k = 1, \ldots, n. \tag{40}$$

It follows that

$$\det\left(\frac{\partial \hat{h}_k \circ A}{\partial q_j}\right) = 2^n n! \prod_{k=1}^{n} e^{2q_k} \prod_{1 \leq i < j \leq n} (e^{2q_j} - e^{2q_i}), \tag{41}$$

and this expression is non-zero for every $q \in \mathcal{C}_n$.

The fact that $A : M \to \hat{M}$ is an action-angle map for the Sutherland system $(M, \omega, H)$ and $A^{-1} : \hat{M} \to M$ is an action-angle map for the RS system $(\hat{M}, \hat{\omega}, \hat{H})$ is expressed by saying that these two many-body systems enjoy 'action-angle duality' relation [12]. In particular, each lives on the action-angle phase space of the other and the position-variables of any of the two systems become action-variables of the other system under the action-angle map.

The general argument of Section 2 now implies directly that any of the commuting Hamiltonians $h_1, \ldots, h_n$, and in particular the Sutherland Hamiltonian $H = \frac{1}{2}h_2$, is maximally superintegrable. Similarly, any of the commuting Hamiltonians $\hat{h}_k$ $(k = 1, \ldots, n)$ of the RS system is

maximally superintegrable. The principal RS Hamiltonian $\hat{H} = \frac{1}{2}(\hat{h}_1 + \hat{h}_{-1})$ can be expressed as a polynomial in terms of $\hat{h}_1, \ldots, \hat{h}_n$, and one can use this to establish its superintegrability as well [1].

At first sight the above reasoning is independent of scattering theory that also could be used to establish maximal superintegrability of the repulsive interactions encoded by $H$ (15) and $\hat{H}$ (20). This is somewhat an illusion, however, since the action-angle maps $A$ and $A^{-1}$ are closely related to the scattering wave maps of the systems under consideration [12]. Nevertheless, an advantage of our arguments is that they do not require any analysis of the large time asymptotic of the dynamics, which is needed in scattering theory. Instead, our reasoning is based on the elegant geometry of the underlying symplectic reduction.

### 3.4 Explicit extra constants of motion in the RS system

The key equation (37) leads to an algebraic algorithm for constructing the maps $A$ and $A^{-1}$ in terms of diagonalization of the Lax matrices $L$ and $\hat{L}$. However, explicit formulae of these action-angle maps are not available. Thus non-trivial effort is required to find extra constants of motion in explicit form both for the rational RS and for the hyperbolic Sutherland system. In the former case, this problem was solved in [1].

The work reported in [1] was inspired by Wojciechowski's paper [18] that explicitly established the superintegrability of the rational Calogero Hamiltonian. In the RS case, since the Lax matrix $\hat{L}$ (18) is positive definite, one can define the smooth real functions

$$\hat{h}_j(\hat{p}, \hat{q}) := \operatorname{tr}\left(\hat{L}(\hat{p}, \hat{q})^j\right), \quad \hat{h}^1_k(\hat{p}, \hat{q}) := \operatorname{tr}\left(\hat{L}(\hat{p}, \hat{q})^k \hat{\boldsymbol{p}}\right), \quad \forall j, k \in \mathbb{Z}. \tag{42}$$

It turned out that these functions satisfy the following Poisson algebra:

$$\{\hat{h}_k, \hat{h}_j\}_{\hat{M}} = 0, \quad \{\hat{h}^1_k, \hat{h}_j\}_{\hat{M}} = j\hat{h}_{j+k}, \quad \{\hat{h}^1_k, \hat{h}^1_j\}_{\hat{M}} = (j-k)\hat{h}^1_{k+j}. \tag{43}$$

The relations (43) were proved in [1] utilizing the symplectic reduction described in Subsection 3.2.

The basic reason for which the (first two) relations of (43) are useful in investigating superintegrability is as follows. Take an *arbitrary* Liouville integrable Hamiltonian

$$\hat{H} = \mathcal{H}(\hat{h}_1, \ldots, \hat{h}_n). \tag{44}$$

Observe that this Hamiltonian Poisson commutes not only with all the $\hat{h}_j$, but also with all functions of the form

$$C^{\hat{H}}_{j,k} := \hat{h}^1_k \{\hat{h}^1_j, \hat{H}\}_{\hat{M}} - \hat{h}^1_j \{\hat{h}^1_k, \hat{H}\}_{\hat{M}}, \qquad \forall j, k \in \mathbb{Z}. \tag{45}$$

Then one should select $(n-1)$ functions out of this set so that together with $\hat{h}_1, \ldots, \hat{h}_n$ they imply the maximal superintegrability of $\hat{H}$. To show functional independence, the selection must use the concrete form of the functions that appear.

As a special case, it was found in [1] that for any fixed $j \in \{1, \ldots, n\}$ the functions

$$C^{\hat{h}_j}_{j,k} = j\hat{h}^1_k \hat{h}_{2j} - j\hat{h}^1_j \hat{h}_{j+k}, \qquad k \in \{1, \ldots, n\} \setminus \{j\} \tag{46}$$

that commute with $\hat{h}_j$ form an independent set together with $\hat{h}_1, \dots, \hat{h}_n$. Furthermore, a set of 'extra constants of motion' that explicitly shows the superintegrability of the RS Hamiltonian $\hat{H} = \frac{1}{2}(\hat{h}_1 + \hat{h}_{-1})$ is provided by

$$\hat{F}_j := \hat{h}^1_j(\hat{h}_2 - n) - \hat{h}^1_1(\hat{h}_{j+1} - \hat{h}_{j-1}), \qquad j = 2, \dots, n. \tag{47}$$

It is worth noting that the quantities $\hat{h}^1_k$ are useful not only for constructing the constants of motion (45), but also since their time development along the solutions $x(t) = (\hat{p}(t), \hat{q}(t))$ of the system $(\hat{M}, \hat{\omega}, \hat{H})$, for any Hamiltonian (44), is especially simple. Namely, since $\{\{\hat{h}^1_k, \hat{H}\}, \hat{H}\}_{\hat{M}} = 0$ follows from (43), we obtain that

$$\hat{h}^1_k(x(t)) = \hat{h}^1_k(x(0)) + t\{h^1_k, \hat{H}\}_{\hat{M}}(x(0)) \tag{48}$$

is linear in time. In this way, $\hat{h}_k$ and $\hat{h}^1_k$ $(k = 1, \dots, n)$ linearize the dynamics. This is similar to the linearization provided by the non-compact analogues of action-angle variables, with the distinctive feature that $\hat{h}_k$ and $\hat{h}^1_k$ are *explicitly* given functions on the phase space.

# 4 Conclusion

In this paper we explained that the hyperbolic Sutherland and the rational RS systems are both maximally superintegrable since Ruijsenaars' duality symplectomorphism [12] between these two systems qualifies as a global action-angle map of maximally non-compact type, and every Liouville integrable system that possesses such action-angle map is maximally superintegrable. Although these results are certainly known to experts, we hope that our self-contained exposition based on the geometric interpretation of the duality [4] may be useful, especially since it can be applied to other examples as well.

Indeed, essentially the same arguments can be applied to the $BC_n$ generalizations of the Sutherland and RS systems, which are encoded by the Hamiltonians

$$\begin{aligned} H_{BC}(q,p) = \frac{1}{2}\sum_{c=1}^n p_c^2 &+ \sum_{1\le a<b\le n}\left(\frac{g^2}{\sinh^2(q_a - q_b)} + \frac{g^2}{\sinh^2(q_a + q_b)}\right) \\ &+ \sum_{c=1}^n \left(\frac{g_1^2}{\sinh^2 q_c} + \frac{g_2^2}{\sinh^2(2q_c)}\right) \end{aligned} \tag{49}$$

and

$$\begin{aligned} \hat{H}_{BC}(\hat{p}, \hat{q}) = \sum_{c=1}^n (\cosh 2\hat{q}_c)\left[1 + \frac{\nu^2}{\hat{p}_c^2}\right]^{\frac{1}{2}}\left[1 + \frac{\chi^2}{\hat{p}_c^2}\right]^{\frac{1}{2}} \prod_{\substack{d=1\\(d\neq c)}}^n \left[1 + \frac{4\mu^2}{(\hat{p}_c - \hat{p}_d)^2}\right]^{\frac{1}{2}}\left[1 + \frac{4\mu^2}{(\hat{p}_c + \hat{p}_d)^2}\right]^{\frac{1}{2}} \\ + \frac{\nu\chi}{4\mu^2}\prod_{c=1}^n\left(1 + \frac{4\mu^2}{\hat{p}_c^2}\right) - \frac{\nu\chi}{4\mu^2}. \end{aligned} \tag{50}$$

The $BC_n$ Sutherland system (49) was introduced by Olshanetsky and Perelomov [8], while the $BC_n$ variant of the RS system (50) is largely due to van Diejen [17]. In a recent work

[11], Pusztai proved by using a suitable symplectic reduction that these two systems are in action-angle duality if their respective 3 coupling parameters are related according to

$$g^2 = \mu^2, \qquad g_1^2 = \frac{1}{2}\nu\chi, \qquad g_2^2 = \frac{1}{2}(\nu - \chi)^2 \tag{51}$$

with arbitrary $\mu^2 > 0$, $\nu > 0$ and $\chi \geq 0$. The duality symplectomorphism is again given by the natural map between two gauge slices, and it yields action-angle maps of maximally non-compact type analogously to the $A_{n-1}$ case.

Finally, we remark that $BC_n$ analogues of the extra constants of motion presented in Subsection 3.4 are still not known, so it could be worthwhile to search for such constants of motion, and to search also for explicit constants of motion in the hyperbolic Sutherland systems.

# A Reduced symplectic form in the Ruijsenaars gauge

The goal of this appendix is to give a self-contained proof of formula (34), which describes the reduced symplectic structure in terms of the Ruijsenaars gauge slice $\hat{S}$ (32). A rather roundabout proof was presented in [4]. Here, we adopt the method of Pusztai [10].

We identify the reduced phase space $P^{\rm red}$ (29) with the global gauge slice $\hat{S}$, whereby the reduced symplectic form becomes

$$\Omega^{\rm red} \equiv \iota_{\hat{S}}^*(\Omega^{\rm ext}). \tag{52}$$

Then, by means of the parametrization of $\hat{S}$ in (32), we regard the components of $\hat{p}$ and $\hat{q}$ as coordinates on $\hat{S}$. Let us denote the Poisson bracket of arbitrary functions $F_1^{\rm red}, F_2^{\rm red} \in C^\infty(\hat{S})$ determined by means on $\Omega^{\rm red}$ as $\{F_1^{\rm red}, F_2^{\rm red}\}$. We wish to find the Poisson brackets

$$\{\hat{p}_\alpha, \hat{p}_\beta\}, \quad \{\hat{p}_\alpha, \hat{q}_\beta\}, \quad \{\hat{q}_\alpha, \hat{q}_\beta\}. \tag{53}$$

As a preparation, we introduce the following functions $\varphi_m, \psi_k \in C^\infty(P^{\rm ext})^K$,

$$\varphi_m(g, J, \zeta) = \frac{1}{2m}{\rm tr}(J^m + (J^\dagger)^m), \qquad \psi_k(g, J, \zeta) = \frac{1}{2}{\rm tr}((J^k + (J^\dagger)^k)g^\dagger Z(\zeta)g), \tag{54}$$

where $m \geq 1$, $k \geq 0$ are integers and

$$Z(\zeta) := ({\rm i}\kappa)^{-1}\zeta + \mathbf{1}_n, \qquad \forall \zeta \in \mathcal{O}_\kappa. \tag{55}$$

It is easily seen that these functions are indeed invariant under the $K$-action (27). We also consider the corresponding reduced functions

$$\varphi_m^{\rm red} := \iota_{\hat{S}}^*(\varphi_m), \qquad \psi_k^{\rm red} := \iota_{\hat{S}}^*(\psi_k). \tag{56}$$

These functions belong to $C^\infty(\hat{S})$ and have the form

$$\varphi_m^{\rm red}(\hat{p}, \hat{q}) = \frac{1}{m}\sum_{j=1}^n \hat{p}_j^m, \qquad \psi_k^{\rm red}(\hat{p}, \hat{q}) = \sum_{j=1}^n \hat{p}_j^k z_j(\hat{p}) e^{-2\hat{q}_j}, \tag{57}$$

with the vector $z(\hat{p})$ defined in (19).

If $F_i^{\rm red} = \iota_{\hat{S}}^*(F_i)$ for some $F_i \in C^\infty(P^{\rm ext})^K$ $(i = 1, 2)$, then the definition of symplectic reduction implies

$$\iota_{\hat{S}}^*(\{F_1, F_2\}^{\rm ext}) = \{F_1^{\rm red}, F_2^{\rm red}\}, \tag{58}$$

where the Poisson bracket on the left-hand-side is computed on $(P^{\rm ext}, \Omega^{\rm ext})$. The idea is to extract the required Poisson brackets in (53) from equality (58) applied to various choices of $F_1, F_2$ from the set of functions $\varphi_m, \psi_k$. Note that $\{F_1, F_2\}^{\rm ext} = \Omega^{\rm ext}(\boldsymbol{X}_{F_2}, \boldsymbol{X}_{F_1})$ with the corresponding Hamiltonian vector fields.

An arbitrary vector field $\boldsymbol{X}$ on $P^{\rm ext}$ (24) can be written as $\boldsymbol{X} = \Delta g \oplus \Delta J \oplus \Delta\zeta$, where at $(g, J, \zeta) \in P^{\rm ext}$ one has $\Delta g \in T_g G$, $\Delta J \in T_J\mathfrak{g} \simeq \mathfrak{g}$ and $\Delta\zeta \in T_\zeta\mathcal{O}_\kappa$. Evaluation of the symplectic form (25) on two vector fields $\boldsymbol{X}$ and $\boldsymbol{X}'$ yields the function

$$\Omega^{\rm ext}(\boldsymbol{X}, \boldsymbol{X}') = \langle g^{-1}\Delta' g, \Delta J\rangle - \langle g^{-1}\Delta g, \Delta' J\rangle + \langle [g^{-1}\Delta' g, g^{-1}\Delta g], J\rangle - \langle \zeta, [D_\zeta, D_\zeta']\rangle, \tag{59}$$

where in the last term we use $\Delta\zeta = [D_\zeta, \zeta]$ and $\Delta'\zeta = [D_\zeta', \zeta]$ with some $\mathfrak{u}(n)$-valued $D_\zeta$ and $D_\zeta'$. It is not difficult to verify the following formulae of the Hamiltonian vector fields of $\varphi_m$ and $\psi_k$:

$$\boldsymbol{X}_{\varphi_m} = gJ^{m-1} \oplus 0 \oplus 0 \quad \text{and} \quad \boldsymbol{X}_{\psi_k} = \Delta g \oplus \Delta J \oplus \Delta\zeta \tag{60}$$

with components

$$\Delta g = g\sum_{j=0}^{k-1} J^j g^\dagger Z(\zeta) g J^{k-1-j}, \tag{61}$$

$$\Delta J = -(J^\dagger)^k g^\dagger Z(\zeta) g - g^\dagger Z(\zeta) g J^k, \tag{62}$$

$$\Delta\zeta = \frac{1}{2\mathrm{i}\kappa}[g(J^k + (J^\dagger)^k)g^\dagger, \zeta]. \tag{63}$$

Note that for $k = 0$ the sum in (61) is vacuous and in this special case $\Delta g = 0$.

**Lemma 1.** *We have $\{\hat{p}_\alpha, \hat{p}_\beta\} = 0$ for all $\alpha, \beta = 1, \dots, n$.*

**Proof.** We readily derive from the above that $\{\varphi_m, \varphi_l\}^{\rm ext} = 0$ for any $m, l \in \mathbb{N}$, which immediately results in $\{\varphi_m^{\rm red}, \varphi_l^{\rm red}\} = 0$. On the other hand, using only the basic properties of the Poisson bracket such as bilinearity and Leibniz rule, we obtain from the formula (57) of these functions that

$$\{\varphi_m^{\rm red}, \varphi_l^{\rm red}\} = \sum_{\alpha,\beta=1}^n \hat{p}_\alpha^{m-1}\{\hat{p}_\alpha, \hat{p}_\beta\}\hat{p}_\beta^{l-1}. \tag{64}$$

Now let us introduce the $n \times n$ matrices $\mathcal{P}_{\alpha,\beta} := \{\hat{p}_\alpha, \hat{p}_\beta\}$ and

$$V_{\alpha,\beta} := \hat{p}_\alpha^{\beta-1}, \qquad \alpha, \beta = 1, 2, \dots, n. \tag{65}$$

Notice that $V$ is a Vandermonde matrix and its determinant is non-zero (as $\hat{p}_1 > \hat{p}_2 > \dots > \hat{p}_n$). Taking $m, l$ from the set $\{1, \dots, n\}$, we can write (65) in matrix form

$$\sum_{\alpha,\beta=1}^n \hat{p}_\alpha^{m-1}\{\hat{p}_\alpha, \hat{p}_\beta\}\hat{p}_\beta^{l-1} = \sum_{\alpha,\beta=1}^n V_{\alpha,m}\mathcal{P}_{\alpha,\beta}V_{\beta,l} = (V^\dagger \mathcal{P} V)_{m,l}. \tag{66}$$

Because this expression must vanish and $V$ is invertible, it follows that $\mathcal{P} = 0$, i.e., $\{\hat{p}_\alpha, \hat{p}_\beta\} = 0$ for all $\alpha, \beta = 1, \dots, n$. *Q.E.D.*

**Lemma 2.** *We have $\{\hat{p}_\alpha, \hat{q}_\beta\} = \delta_{\alpha,\beta}$ for all $\alpha, \beta = 1, \dots, n$.*

**Proof.** Taking arbitrary

$$k = 0, 1, \dots, n-1 \quad \text{and} \quad l = 1, \dots, n, \tag{67}$$

it can be checked that $\{\psi_k, \varphi_l\}^{\text{ext}} = 2\psi_{k+l-1}$ holds at all triples $(g, J, \zeta)$ for which $J = J^\dagger$. Hence we must have

$$\{\psi_k^{\text{red}}, \varphi_l^{\text{red}}\} = 2\psi_{k+l-1}^{\text{red}}. \tag{68}$$

Using the basic properties of the Poisson bracket and the statement of Lemma 1, we can directly calculate this Poisson bracket as

$$\{\psi_k^{\text{red}}, \varphi_l^{\text{red}}\} = \sum_{\alpha=1}^n \hat{p}_\alpha^k z_\alpha e^{-2\hat{q}_\alpha} \sum_{\beta=1}^n \{-2\hat{q}_\alpha, \hat{p}_\beta\} \hat{p}_\beta^{l-1}. \tag{69}$$

The comparison of the last two equations leads to

$$\sum_{\alpha=1}^n \hat{p}_\alpha^k z_\alpha e^{-2\hat{q}_\alpha} \left( \sum_{\beta=1}^n \{-2\hat{q}_\alpha, \hat{p}_\beta\} \hat{p}_\beta^{l-1} - 2\hat{p}_\alpha^{l-1} \right) = 0. \tag{70}$$

By introducing the $n \times n$ matrix

$$\mathcal{A}_{\alpha,\beta} = z_\alpha e^{-2\hat{q}_\alpha} \left( \sum_{\gamma=1}^n \{-2\hat{q}_\alpha, \hat{p}_\gamma\} \hat{p}_\gamma^{\beta-1} - 2\hat{p}_\alpha^{\beta-1} \right) \tag{71}$$

we can write (71) as $(V^\dagger \mathcal{A})_{k+1,l} = 0$. Since $V$ (65) is invertible, we conclude that $\mathcal{A} = 0$. Now if we collect the expressions $\{-2\hat{q}_\alpha, \hat{p}_\beta\}$ in the $n \times n$ matrix $\mathcal{B}_{\alpha,\beta} := \{-2\hat{q}_\alpha, \hat{p}_\beta\}$, then the vanishing of $\mathcal{A}$ can be re-stated as the matrix equation $\mathcal{B}V - 2V = 0$. This entails that $\mathcal{B} = 2\mathbf{1}_n$, which is equivalent to $\{\hat{p}_\alpha, \hat{q}_\beta\} = \delta_{\alpha,\beta}$ for all $\alpha, \beta$. *Q.E.D.*

**Lemma 3.** *We have $\{\hat{q}_\alpha, \hat{q}_\beta\} = 0$ for all $\alpha, \beta = 1, \dots, n$.*

**Proof.** We now determine the reduced Poisson bracket

$$\{\psi_k^{\text{red}}, \psi_l^{\text{red}}\}, \qquad \forall k, l = 0, 1, \dots, n-1, \tag{72}$$

in two ways. First we use $\{\psi_k^{\text{red}}, \psi_l^{\text{red}}\} = \{\psi_k, \psi_l\}^{\text{ext}} \circ \iota_{\hat{S}}$ and obtain by calculating the right-hand-side that

$$\{\psi_k^{\text{red}}, \psi_l^{\text{red}}\} = -2(k-l) \sum_{\alpha=1}^n \hat{p}_\alpha^{k+l-1} z_\alpha^2 e^{-4\hat{q}_\alpha} - 16\kappa^2 \sum_{\substack{\alpha,\beta=1 \\ (\alpha \neq \beta)}}^n \frac{\hat{p}_\alpha^k \hat{p}_\beta^l z_\alpha z_\beta e^{-2(\hat{q}_\alpha + \hat{q}_\beta)}}{(4\kappa^2 + (\hat{p}_\alpha - \hat{p}_\beta)^2)(\hat{p}_\alpha - \hat{p}_\beta)}. \tag{73}$$

Then direct calculation of $\{\psi_k^{\rm red}, \psi_l^{\rm red}\}$, utilizing basic properties of the Poisson bracket together with the preceding lemmas, gives

$$\begin{aligned}\{\psi_k^{\rm red}, \psi_l^{\rm red}\} = 2\sum_{\alpha,\beta=1}^n \left[\hat{p}_\alpha^k z_\alpha \frac{\partial \hat{p}_\beta^l z_\beta}{\partial \hat{p}_\alpha} - \hat{p}_\beta^k z_\beta \frac{\partial \hat{p}_\alpha^k z_\alpha}{\partial \hat{p}_\beta}\right] e^{-2(\hat{q}_\alpha+\hat{q}_\beta)} \\ + 4\sum_{\alpha,\beta=1}^n \hat{p}_\alpha^k \hat{p}_\beta^l z_\alpha z_\beta e^{-2(\hat{q}_\alpha+\hat{q}_\beta)}\{\hat{q}_\alpha, \hat{q}_\beta\}.\end{aligned} \tag{74}$$

Simple algebraic manipulations permit to spell this out more explicitly

$$\begin{aligned}\{\psi_k^{\rm red}, \psi_l^{\rm red}\} = -2(k-l)\sum_{\alpha=1}^n \hat{p}_\alpha^{k+l-1} z_\alpha^2 e^{-4\hat{q}_\alpha} - 16\kappa^2 \sum_{\substack{\alpha,\beta=1 \\ (\alpha\neq\beta)}}^n \frac{\hat{p}_\alpha^k \hat{p}_\beta^l z_\alpha z_\beta e^{-2(\hat{q}_\alpha+\hat{q}_\beta)}}{(4\kappa^2 + (\hat{p}_\alpha - \hat{p}_\beta)^2)(\hat{p}_\alpha - \hat{p}_\beta)} \\ + 4\sum_{\alpha,\beta=1}^n \hat{p}_\alpha^k \hat{p}_\beta^l z_\alpha z_\beta e^{-2(\hat{q}_\alpha+\hat{q}_\beta)}\{\hat{q}_\alpha, \hat{q}_\beta\}.\end{aligned} \tag{75}$$

Comparing equations (73) and (75), we then find that

$$\sum_{\alpha,\beta=1}^n \hat{p}_\alpha^k \hat{p}_\beta^l z_\alpha z_\beta e^{-2(\hat{q}_\alpha+\hat{q}_\beta)}\{\hat{q}_\alpha, \hat{q}_\beta\} = 0. \tag{76}$$

Inspecting this equation using the non-degeneracy of the matrix $V$ (65) and that the functions $z_\alpha$ never vanish, we find that $\{\hat{q}_\alpha, \hat{q}_\beta\}$ must vanish for all $\alpha$ and $\beta$. *Q.E.D.*

The three lemmas together prove the important formula (34), which was proved in [4] by a less self-contained method.

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